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Fractional Integration and Differentiation of Hyper-Geometric Function for Power Function

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Abstract

In This paper, we have to convert hyper-geometric function into hyper-geometric function for power function. The applications of hyper-geometric function in a various field of physical and applied science are demonstrated, the success of the application of hyper-geometric function in many areas of science and engineering. So, the function and its properties are useful for solving the problems in physics, biology and science.

Keywords: *Fractional calculus operators, Hyper-geometric function, Special functions, Mathematics Subject Classification-26A33, 33C60, 44A15.*

I. Introduction

Hyper-geometric function for power function is a particular case of hyper-geometric series as in [7]. A hyper-geometric series with p upper parameters $a_1, a_2, \ldots\ldots a_p$ and q lower parameters $b_1, b_2, \ldots\ldots b_q$ is denoted and defined by

$$ {}_pF_q\left(a_1 \ldots . a_p, b_1 \ldots . b_q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots\ldots (a_p)_k}{(b_1)_k \ldots\ldots (b_q)_k} \frac{z^k}{k!} \qquad (1) $$

Here $(a_j)_k\,(b_j)_k$ are pochammer symbols. Where

$(a_j)_k = a_j(a_j+1)\ldots(a_j+k-1),\ (a_j)_0 = 1,\ a_j \neq 0.$

II. Definition

Firstly, we give the definition of hyper-geometric function for power function, introduced by the author

$$ {}_pF_q\left(a_1 \ldots . a_p, b_1 \ldots . b_q; z^m\right) = {}_pF_q(z^m) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots\ldots (a_p)_k}{(b_1)_k \ldots\ldots (b_q)_k} \frac{z^{mk}}{k!} \qquad (2) $$

Here $(a_j)_k\,(b_j)_k$ are the Pochammer symbols and $m > 0$. If the parameter a_j is a negative integer and if no b_j is negative integer or zero then the series (2) terminates into polynomials. If b_j ; $j = 1,2,3 \ldots\ldots\ldots q$ is a negative integer or zero then the series (2) does not make sense unless have is an a_j; $j = 1,2,3 \ldots\ldots\ldots p$ such that $(a_j)_k = 0$ before $(b_j)_k = 0$. Using a ratio test, it is evident that the series (2) is convergent for every z, if $p \le q$,it is convergent for, when $p = q + 1$ and divergent, when $p > q + 1$. If $p = q + 1$ and $|z| = 1$, then the series can converge in some case. We take,

$$ \beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j $$

We see that, when $p = q + 1$ the series is absolutely convergent for $|z| = 1$ if $R(\beta) < 0$, convergent for $z = -1$, $0 \le R(\beta) < 1$and divergent for $|z| = 1, 1 \le R(\beta)$.

Some special case of ${}_pF_q(z^m)$ function:

A) ${}_0F_0$ i.e. no upper or lower parameter and $m = 1$.

$$ {}_0F_0(\;;\;;\pm z) = \sum_{k=0}^{\infty} \frac{(\pm z)^k}{k!} = e^{\pm z} \quad (3)$$

Hence ${}_0F_0$ is reduced to the exponential series.

B) ${}_1F_0$ i.e. one upper parameter α and no lower parameter.

If α is negative then the series terminates into polynomials and in the case, the condition, $|z| < 1$, the series convert into binomial series.

$$ {}_1F_0(\alpha\;;\;;\pm z^m) = \sum_{k=0}^{\infty} (\alpha)_k \frac{z^{mk}}{k!} = (1 - z^m)^k \quad (4)$$

For $|z| < 1$

Thus, it is the binomial series as in [8].

III. Fractional Integral and Fractional Derivative of the Hyper-Geometric Function for Power Function

Let us consider the fractional Riemann – Liouville (R-L) integral operator, as in [7] (for lower limit $a = 0$ with respect to variable z) of the hyper-geometric function (2).

$$ I_z^v \, {}_pF_q\,(z^m) = \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} \, {}_pF_q\,(t^m)\, dt $$

$$ = \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k}{(b_1)_k \, \ldots \, (b_q)_k} \frac{t^{mk}}{k!}\, dt $$

$$ = \frac{1}{\Gamma(v)} \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k}{(b_1)_k \, \ldots \, (b_q)_k} \frac{1}{k!} \int_0^z (z-t)^{v-1}\, t^{mk}\, dt $$

$$ = \frac{1}{\Gamma(v)} \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k}{(b_1)_k \, \ldots \, (b_q)_k} \frac{1}{k!} z^{mk+1+v-1} B(mk+1, v) $$

$$ = \frac{1}{\Gamma(v)} \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k}{(b_1)_k \, \ldots \, (b_q)_k} \frac{1}{k!} z^{mk+v} \frac{\Gamma(mk+1)\Gamma(v)}{\Gamma(mk+1+v)} $$

$$ = z^v \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k}{(b_1)_k \, \ldots \, (b_q)_k} \frac{z^{mk}}{k!} \frac{\Gamma(mk+1)}{\Gamma(mk+1+v)} $$

$$ = z^v \frac{\Gamma(mk+1-k)}{\Gamma(mk+1+v-k)} $$

$$ \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k \,(mk+1-k)_k}{(b_1)_k \, \ldots \, (b_q)_k (mk+1-k)_k} \frac{z^{mk}}{k!} $$

$$ I_z^v \, {}_pF_q\,(z^m) = \frac{\Gamma(mk+1-k)}{\Gamma(mk+1+v-k)} z^v $$

$$ {}_pF_q\left(a_1 \ldots a_p, (mk+1-k), b_1 \ldots b_q, (mk+1-k); z^m\right) \quad (5)$$

$R-L$ Fractional derivative of Hyper-geometric Function which indices p,q are increased to $(p+1)(q+1)$.

Analogously, $R-L$fractional derivative operator as in [7] of the Hyper-geometric Function with respect to z.

$$ D_z^v \, {}_pF_q\,(z^m) = \frac{1}{\Gamma(n-v)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-v-1} \, {}_pF_q\,(t^m)\, dt $$

$$ = \frac{1}{\Gamma(n-v)} \left(\frac{d}{dz}\right)^n \int_0^z (z-t)^{n-v-1} $$

$$ \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k}{(b_1)_k \, \ldots \, (b_q)_k} \frac{t^{mk}}{k!}\, dt $$

$$ = \frac{1}{\Gamma(n-v)} \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k}{(b_1)_k \, \ldots \, (b_q)_k} \left(\frac{d}{dz}\right)^n \frac{1}{k!} $$

$$ \int_0^z (z-t)^{n-v-1}\, t^{mk}\, dt $$

$$ = \frac{1}{\Gamma(n-v)} \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k}{(b_1)_k \, \ldots \, (b_q)_k} $$

$$ \left(\frac{d}{dz}\right)^n \frac{1}{k!} z^{mk+n-v} B(mk+1, n-v) \quad (6)$$

We use the modified Beta-function

$$ \int_a^b (b-t)^{\beta-1}\,(t-a)^{\alpha-1}\, dt = (b-a)^{\alpha+\beta-1} B(\alpha,\beta), $$

$$ \text{for } R(\alpha) > 0, R(\beta) > 0 $$

$$ = \frac{1}{\Gamma(n-v)} \sum_{k=0}^{\infty} \frac{(a_1)_k \, \ldots \, (a_p)_k}{(b_1)_k \, \ldots \, (b_q)_k} \left(\frac{d}{dz}\right)^n $$

$$ \frac{1}{k!} z^{mk+n-v} \frac{\Gamma(mk+1)\Gamma(n-v)}{\Gamma(mk+1+n-v)} \quad (7)$$

Differentiation n times the term z^{mk+n-v} and using again $\Gamma(a+k) = (a)_k \Gamma(a)$, equation (7) reduces to

$$= \frac{1}{\Gamma(n-v)} \sum_{k=o}^{\infty} \frac{(a_1)_k \;.....(a_p)_k}{(b_1)_k \;.....(b_q)_k} \frac{(mk+n-v)!}{\Gamma(k+1)}$$

$$z^{mk-v} \frac{\Gamma(mk+1)\Gamma(n-v)}{\Gamma(mk+1+n-v)}$$

$$= z^{-v}\Gamma(mk+1-k) \sum_{k=o}^{\infty} \frac{(a_1)_k \;.....(a_p)_k}{(b_1)_k \;.....(b_q)_k (1)_k}$$

$$(mk+1-k)_k z^{mk}$$

$$D_z^v \,{}_pF_q\,(z^m) = \Gamma(mk+1-k)\, z^{-v}$$

$${}_pF_q\left(a_{1},\ldots,a_{p},(mk+1-k),b_{1},\ldots,b_{q},(1);z^m\right) \qquad (8)$$

$(mk+1) > 0$, gives a $R-L$ hypergeometric function of hypergeometric function for Power function, which indices p, q are increased to (p+1),(q+1).

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